On the Impact of Robust Statistics on Imprecise Probability Models

A Review

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Imprecise Probabilities

Classical probability theory:

- Probabilities specified by exact real numbers $P(A)$
- “The probability of an event $A$ is $P(A) = 0.4281635907\ldots$”
Imprecise Probabilities

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- Probabilities specified by exact real numbers $P(A)$
- "The probability of an event $A$ is $P(A) = 0.4281635907\ldots""

Imprecise probabilities:

- Probabilities given by lower $L(A)$ and upper bounds $U(A)$
- "The probability of an event $A$ lies between $L(A) = 0.38$ and $U(A) = 0.45""
Imprecise Probabilities: Sets of Probability Measures

Classical probability theory:

- A single probability measure

\[ P : A \mapsto P(A) \]
Imprecise Probabilities: Sets of Probability Measures

**Classical probability theory:**

- A single probability measure

\[ P : A \mapsto P(A) \]

**Imprecise probabilities:**

- A set of possible probability measures

\[ \mathcal{M} = \left\{ P : A \mapsto P(A) \mid L(A) \leq P(A) \leq U(A) ~ \forall A \right\} \]

\( \mathcal{M} \) is called *structure*. 
Imprecise Probabilities and Robust Statistics

**Imprecise Probabilities**

- Slightly different concepts; developed only recently
  - Walley (1991): coherent lower/upper previsions
  - Weichselberger (2001): interval probabilities
  - ...

- Generalization of classical probabilities
  - model uncertainties about exact, true probabilities

- Successfully applied to many engineering problems
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- Tries to deal with (small) deviations from modeling assumptions in statistics
- Robust statistics hardly received attention in engineering science
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Close relationship between both areas
Classical Statistics

▶ A known precise statistical model \( \{ P_\theta | \theta \in \Theta \} \) is assumed
  ▶ \( \theta \): an unknown parameter
  ▶ \( P_\theta \): a probability measure depending on the unknown \( \theta \)
  ▶ True probabilities \( P_\theta(A) \) exactly known except for \( \theta \)
▶ Example: \( P_\theta \) is the normal distribution \( N(\theta, 1) \)
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  $\rightarrow$ not correct
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  \[ \text{not correct, quite often} \]
- strict assumptions lead to unreliable conclusions
Robust Statistics

- \( \{P_\theta | \theta \in \Theta \} \) is a known precise statistical model
- \( \{P_\theta | \theta \in \Theta \} \) is assumed to be approximately true:
  - \( U(P_\theta) \): neighborhood about \( P_\theta \)
  - It is possible that
    - the true distribution \( P \neq P_\theta \)
  - But: The true distribution \( P \) lies in the neighborhood about \( P_\theta \)
    - \( P \in U(P_\theta) \)
- Goal: Develop statistical procedures which are still reliable
Parametric Model $\{P_\theta | \theta \in \Theta\}$
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Robust Statistics and Imprecise Probabilities

- $\{P_\theta | \theta \in \Theta\}$ is a known precise statistical model
- Robust statistics uses neighborhoods $U(P_\theta)$ about $P_\theta$

**Theorem:** Neighborhoods $U(P_\theta)$ are structures of imprecise probabilities.
Robust Statistics and Imprecise Probabilities

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→ Models used in robust statistics are special cases of imprecise probabilities
Robust Statistics and Imprecise Probabilities

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→ Models used in robust statistics are special cases of imprecise probabilities

**Goal:**

Robust Statistics \hspace{2cm} Imprecise Probabilities

statistical procedures \hspace{2cm} generalize
Hypothesis Testing

Classical Statistics:

\[ H_0 : P = P_0 \quad \text{vs.} \quad H_1 : P = P_1 \]
Hypothesis Testing

Classical Statistics:

\[ H_0 : \ P = P_0 \quad \text{vs.} \quad H_1 : \ P = P_1 \]

Robust Statistics:

\[ H_0 : \ P \in U(P_0) \quad \text{vs.} \quad H_1 : \ P \in U(P_1) \]
Hypothesis Testing

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\[ H_0 : \ P = P_0 \quad \text{vs.} \quad H_1 : \ P = P_1 \]

Robust Statistics:

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Imprecise Probabilities:

\[ H_0 : \ P \in \mathcal{M}_0 \quad \text{vs.} \quad H_1 : \ P \in \mathcal{M}_1 \]
Hypothesis Testing

Classical Statistics:

Robust Statistics:

Imprecise Probabilities:
Hypothesis Testing: Least Favorable Pairs

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Imprecise Probabilities:
Hypothesis Testing: Least Favorable Pairs

Classical Statistics:

\[
P_0 \quad P_1
\]

Robust Statistics:

\[
U(P_0) \quad U(P_1)
\]

Imprecise Probabilities:

\[
M_0 \quad M_1
\]
Estimation

Classical statistics

- “optimal” estimators available
- usually: “optimal” estimators are unreliable
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Robust statistics

- “optimal” robust estimators available; see e.g. Rieder (1994) and Kohl et al. (2009)
  - fix the amount of robustness/reliability you want to have
  - choose the most efficient estimator under this robustness constraint
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Imprecise probabilities

▶ general estimation problems hardly considered explicitly
▶ “optimal” estimators not available so far
→ generalize theory of robust estimating to imprecise probabilities
References


The handout to this talk is also available on my homepage

http://www.staff.uni-bayreuth.de/~btms04/index.html